Second and Higher Order Linear Differential Equations

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Seminar in Engineering Analysis

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California State University Northridge

Outline

- Review last class and homework
- Apply material from last class to mechanical vibrations
- Higher order equations with constant coefficients
  - Homogenous and nonhomogenous solutions
- Existence and uniqueness of solutions for higher order equations

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Review Undetermined Coefficients

- Used for constant coefficient equation y" + ay' + by = r(x)
- Solution is $y = y_P + y_H$, where y_H is solution of $y_H'' + ay_H' + by = 0$
- Postulate a solution for y_P following guidelines on next two charts
- Plug solution into ODE and solve for unknown coefficients
  - Overall coefficients of like terms on both sides of ODE must vanish

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Table of Trial y_P Solutions

For these r(x)	Start with this y_P
$r(x) = Ae^{ax}$	$y_P = Be^{ax}$
$r(x) = Ax^n$	$y_P = a_0 + a_1x + \dots + a_nx^n$
$r(x) = A\sin \omega t$	$y_P = B \sin \omega t + C \cos \omega t$
$r(x) = A\cos \omega t$	
$r(x) = Ae^{ax}\sin \omega t$	$y_P = e^{ax} (B \sin \omega t + C \cos \omega t)$
$r(x) = Ae^{ax}\cos \omega t$	

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Special Rules

- If the right-hand-side, r(x) consists of more than one term from the previous table, use a y_P that contains all the corresponding y_P terms
  - For $r(x) = A\cos bx + Ce^{dx}$, use $y_P = E \sin bx + F \cos bx + Ge^{dx}$
- If r(x) is proportional to a solution for the homogenous equation, use y_P equal to x times the y_P shown in the table
  - For a double root, multiply table y_P by x^2

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Review Parameter Variation

- Want to solve linear equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

- Have to solve homogenous equation to get two (LI) solutions y_1 and y_2
- Define W from these two solutions

$$W = y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}$$

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Review Parameter Variation II

- Define u(x) and v(x) such that $y_P = y_1u + y_2v$, where u and v are found from the following integrals

$$u = -\int \frac{y_2 r(x)}{W(x)} dx \qquad v = \int \frac{y_1 r(x)}{W(x)} dx$$

$$y_P = y_1 u + y_2 v = -y_1 \int \frac{y_2 r(x)}{W(x)} dx + y_2 \int \frac{y_1 r(x)}{W(x)} dx$$

- Get $y = y_H + y_P$ and evaluate constants in y_H solution from initial conditions

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Nonhomogenous Summary

- Undetermined coefficients is simpler approach but is limited
  - Constant coefficient equations
  - Limited set of functions
- Variation of parameters is more complex, but handles more cases
- In reality, there are no general methods to get homogenous solution to linear, second-order ODE without constant coefficients

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Higher Order Equations

- General n^{th} order linear equation

$$\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y = r(x)$$

- Treatment similar to second order
- Look at homogenous solution first
- Combine with particular solution
- Must consider ODE with constant coefficients to get any general results
  - This is similar to second order

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Higher Order Equations II

- Look at general n^{th} order differential equation with constant coefficients

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = r(x)$$

- Find y_H from homogenous ODE

$$\frac{d^n y_H}{dx^n} + a_{n-1}\frac{d^{n-1} y_H}{dx^{n-1}} + \cdots + a_1\frac{dy_H}{dx} + a_0 y_H = 0$$

- Homogenous solution $y_H = \sum_{k=1}^{n} C_k e^{\lambda_k x}$

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Higher Order Equations III

- Homogenous solution $y_H = \sum_{k=1}^{n} C_k e^{\lambda_k x}$
- In homogenous solution, the values of λ_k are solutions to the equation $\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0$
- Complex solutions occur as complex conjugates giving sines and cosines
- For double roots λ_k = DR, we modify solution to give $(C_k + C_{k+1}x)\, e^{(DR)x}$

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Higher Order Equations IV

- For nonhomogenous equations we can find the total solution $y = y_H + y_P$
- y_P may be found by undetermined coefficients or variation of parameters
  - Use same process for method of undetermined coefficients
  - Variation of parameters is more complex since it involves solution of simultaneous equations for new solutions

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Higher Order Equations V

- There are n linearly-independent solutions to a linear, homogenous n^{th} order ODE
- The n linearly-independent solutions form a basis for all solutions
  – Use same process for method of undetermined coefficients
  – Variation of parameters is more complex since it involves solution of simultaneous equations for new solutions

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Existence and Uniqueness

- General linear, homogenous, n^{th} order ODEs have a unique solution over a < x < b if all the $p_k(x)$ are continuous there

$$\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y = 0$$

- The proposed solutions $y_k(x)$ to the homogenous ODE are linearly independent if the Wronskian (see next chart) is nonzero

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Wronski Determinant

- Wronskian, W, for nth order ODE (with notation that $y^{(k)}$ denotes d^ky/dx^k)

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & y_3^{(1)} & \cdots & \cdots & y_n^{(1)} \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} & \cdots & \cdots & y_n^{(2)} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \cdots & \cdots & y_n^{(n)} \end{vmatrix}$$

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Application: Structural Member

- An elastic beam with an applied load, f(x), per unit length, in the y direction (normal to the beam)
  – Beam is bent under this load
  – Bending moment, M(x) is given by second-order ODE: $d^2M/dx^2 = f(x)$
  – Final deflection is d^2y/dx^2
  – M = EI d^2y/dx^2 where E is Young's modulus and I is moment of inertia

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Structural Member ODE

- Combine $d^2M/dx^2 = f(x)$ and M = EI d^2y/dx^2 to get $EId^4y/dx^4 = f(x)$
  – SI units for these quantities are meters for x and y, N/m^2 for E, m^4 for I, N/m for f(x), and N·m for M
  – dimensions for n^{th} order derivative are dimensions of numerator divided by $(\text{denominator dimensions})^n$
  – Have a total of four boundary conditions at x = 0 and x = L
  – Equation has separable solution

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Solving the Equation

$$\frac{d^4y}{dx^4} = \frac{f(x)}{EI} \quad \Rightarrow \quad \frac{d^3y}{dx^3} = \frac{1}{EI}\int f(x)dx + C_1$$

$$\frac{d^3y}{dx^3} = \frac{1}{EI}\int\left(\int f(x)dx\right)dx + C_1x + C_2$$

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\int\left[\int\left(\int f(x)dx\right)dx\right]dx + C_1x^2 + C_2x + C_3$$

$$\frac{dy}{dx} = \frac{1}{EI}\int\left\{\int\left[\int\left(\int f(x)dx\right)dx\right]dx\right\}dx + C_1x^3 + C_2x^2 + C_3x + C_4$$

- Apply boundary conditions to find constants of integration

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Application: Forced Vibrations

- Last week we showed solutions for free vibrations of spring-mass-damper system
- ODE was $md^2y/dt^2 + cdy/dt + ky = 0$
- Imposed force gives nonhomogenous ODE $md^2y/dt^2 + cdy/dt + ky = f(t)$
- Consider example where $f(t) = F_0 \cos \omega t$
- Undetermined coefficient trial solution is $y_P = A \sin \omega t + B \cos \omega t$

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Forced Vibrations II

- Derivatives of $y_P = A \sin \omega t + B \cos \omega t$
- $y_P' = \omega A \cos \omega t - \omega B \sin \omega t$
- $y_P'' = -\omega^2 A \sin \omega t - \omega^2 B \cos \omega t$
- Substitute into ODE: $md^2y/dt^2 + cdy/dt + ky = F_0 \cos \omega t$
- $m[-\omega^2 A \sin \omega t - \omega^2 B \cos \omega t]$
  $+ c[\omega A \cos \omega t - \omega B \sin \omega t]$
  $+ k[A \sin \omega t + B \cos \omega t] = F_0 \cos \omega t$

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Forced Vibrations III

- Rearrange to collect sines and cosines
- $m[-\omega^2 A \sin \omega t - \omega^2 B \cos \omega t]$
  $+ c[\omega A \cos \omega t - \omega B \sin \omega t]$
  $+ k[A \sin \omega t + B \cos \omega t] = F_0 \cos \omega t$
- $[-m\omega^2 A - c\omega B + kA] \sin \omega t +$
  $[-m\omega^2 B + c\omega A + kB] \cos \omega t = F_0 \cos \omega t$
- Equate coefficients of sine and cosine terms on both sides of the equation

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Forced Vibrations IV

- $[-m\omega^2 A - c\omega B + kA] \sin \omega t + [-m\omega^2 B + c\omega A + kB] \cos \omega t = F_0 \cos \omega t$
- $(-m\omega^2 + k)A - \quad c\omega B = 0$ (sine terms)
- $\quad c\omega A - (-m\omega^2 + k)B = F_0$ (cosines)
- Cramer's rule solution gives

$$A = \frac{\begin{vmatrix} 0 & -\omega c \\ F_0 & k - m\omega^2 \end{vmatrix}}{\begin{vmatrix} k - m\omega^2 & -\omega c \\ \omega c & k - m\omega^2 \end{vmatrix}} \qquad B = \frac{\begin{vmatrix} k - m\omega^2 & 0 \\ \omega c & F_0 \end{vmatrix}}{\begin{vmatrix} k - m\omega^2 & -\omega c \\ \omega c & k - m\omega^2 \end{vmatrix}}$$

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Define $\omega_0^2 = k/m$ in Solution

$$A = \frac{\begin{vmatrix} 0 & -\omega c \\ F_0 & k - m\omega^2 \end{vmatrix}}{\begin{vmatrix} k - m\omega^2 & -\omega c \\ \omega c & k - m\omega^2 \end{vmatrix}} \qquad B = \frac{\begin{vmatrix} k - m\omega^2 & 0 \\ \omega c & F_0 \end{vmatrix}}{\begin{vmatrix} k - m\omega^2 & -\omega c \\ \omega c & k - m\omega^2 \end{vmatrix}}$$

$$A = \frac{F_0 \omega c}{(k - m\omega^2)^2 + \omega^2 c^2} \qquad B = \frac{F_0 (k - m\omega^2)}{(k - m\omega^2)^2 + \omega^2 c^2}$$

$$A = \frac{F_0 \omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \qquad B = \frac{m F_0 (\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$$

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Undamped Case, c = 0

$$A = \frac{F_0 \omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} = 0$$

$$B = \frac{m F_0 (\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} = \frac{m F_0 (\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2} = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

- $y_P = A \sin \omega t + B \cos \omega t = B \cos \omega t$
- From last week, $y_H = C \sin \omega_0 t + D \cos \omega_0 t = E \cos(\omega_0 t + \delta)$
- Look at initial conditions

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Undamped Case II

$$y = y_H + y_P = C\sin\omega_0 t + D\cos\omega_0 t + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\cos\omega t$$

- Initial conditions y(0) = y_0 and y'(0) = v_0
- $y_0 = D + F_0/m/(\omega_0^2 - \omega^2)$
- $v_0 = \omega_0 C$

$$y = \frac{v_0}{\omega_0}\sin\omega_0 t + \left[y_0 - \frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\right]\cos\omega_0 t + \frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t$$

$$\frac{y}{y_0} = \frac{v_0}{\omega_0 y_0}\sin\omega_0 t + \left[1 - \frac{F_0}{m y_0\left(\omega_0^2-\omega^2\right)}\right]\cos\omega_0 t + \frac{F_0}{m y_0\left(\omega_0^2-\omega^2\right)}\cos\left(\frac{\omega}{\omega_0}\omega_0 t\right)$$

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Undamped Case III

$$\frac{F_0}{m y_0\left(\omega_0^2-\omega^2\right)} = \frac{F_0}{m y_0 \omega_0^2\left[1-\left(\omega/\omega_0\right)^2\right]} = \frac{F_0}{m y_0 \frac{k}{m}\left[1-\left(\omega/\omega_0\right)^2\right]}$$

$$\frac{y}{y_0} = \frac{v_0}{\omega_0 y_0}\sin\omega_0 t + \left[1 - \frac{F_0/ky_0}{\left(1-\omega^2/\omega_0^2\right)}\right]\cos\omega_0 t + \frac{F_0/ky_0}{\left(1-\omega^2/\omega_0^2\right)}\cos\left(\frac{\omega}{\omega_0}\omega_0 t\right)$$

- y/y_0 is a function of $\omega_0 t$ and the following three (dimensionless) parameters: $v_0/\omega_0 y_0$, F_0/ky_0 and ω/ω_0

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Undamped Case IV

- Start with solution below and convert $\omega_0 t$ sine and cosine terms to a cosine term

$$y = A\sin\omega_0 t + B\cos\omega_0 t + \frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t$$

$$A = \frac{v_0}{\omega_0} \qquad B = y_0 - \frac{F_0}{m\left(\omega_0^2-\omega^2\right)}$$

$$y = C\cos\left(\omega_0 t - \delta\right) + \frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t \qquad \delta = \tan^{-1}\left(\frac{A}{B}\right)$$

$$C = \sqrt{A^2+B^2} = \sqrt{\left(\frac{v_0}{\omega_0}\right)^2 + \left(y_0 - \frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\right)^2}$$

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Undamped Case V

- Compute dimensionless C and δ

$$\frac{C}{y_0} = \sqrt{\left(\frac{v_0}{y_0\omega_0}\right)^2 + \left(1 - \frac{F_0}{y_0 m\left(\omega_0^2-\omega^2\right)}\right)^2}$$

$$\delta = \tan^{-1}\left(\frac{A}{B}\right) = \tan^{-1}\left(\frac{\dfrac{v_0}{\omega_0}}{y_0 - \dfrac{F_0}{m\left(\omega_0^2-\omega^2\right)}}\right) = \tan^{-1}\left(\frac{\dfrac{v_0}{y_0\omega_0}}{1 - \dfrac{F_0}{m y_0\left(\omega_0^2-\omega^2\right)}}\right)$$

$$\frac{y}{y_0} = \frac{C}{y_0}\cos\left(\omega_0 t - \delta\right) + \frac{F_0}{y_0 m\left(\omega_0^2-\omega^2\right)}\cos\omega t$$

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Undamped Case VI

- y/y_0 is a function of $\omega_0 t$, $v_0/\omega_0 y_0$, F_0/ky_0, and ω/ω_0

$$\frac{F_0}{m y_0\left(\omega_0^2-\omega^2\right)} = \frac{F_0/ky_0}{1-\left(\omega/\omega_0\right)^2}$$

$$\frac{C}{y_0} = \sqrt{\left(\frac{v_0}{y_0\omega_0}\right)^2 + \left(1 - \frac{F_0/ky_0}{1-\left(\omega/\omega_0\right)^2}\right)^2} \qquad \tan\delta = \frac{v_0/y_0\omega_0}{1 - \dfrac{F_0/ky_0}{1-\left(\omega/\omega_0\right)^2}}$$

$$\frac{y}{y_0} = \frac{C}{y_0}\cos\left(\omega_0 t - \delta\right) + \frac{F_0/ky_0}{1-\left(\omega/\omega_0\right)^2}\cos\left(\frac{\omega}{\omega_0}\omega_0 t\right)$$

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Zero Initial Conditions

- Without forcing (F_0 = 0), when $y_0 = v_0$ = 0, the solution is y = 0 for all t
- Forcing gives a nonzero solution
- Start with general solution for c = 0

$$y = y_H + y_P = C\sin\omega_0 t + D\cos\omega_0 t + \frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\cos\omega t$$

- v_0 = y'(0) = 0 gives C = 0
- y_0 = y(0) = 0 gives $D = -F_0/m/(\omega_0^2 - \omega^2)$

$$y = \frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\left(\cos\omega t - \cos\omega_0 t\right)$$

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Zero Initial Conditions II

- Rearrange solution for c = $y_0 = v_0 = 0$

$$y = \frac{F_0}{m\omega_0^2\left(1-\omega^2/\omega_0^2\right)}\left[\cos\left(\frac{\omega}{\omega_0}\omega_0 t\right) - \cos\omega_0 t\right]$$

$$\frac{ym\omega_0^2}{F_0} = \frac{ym}{F_0}\frac{k}{m} = \frac{yk}{F_0} = \frac{1}{\left(1-\omega^2/\omega_0^2\right)}\left[\cos\left(\frac{\omega}{\omega_0}\omega_0 t\right) - \cos\omega_0 t\right]$$

- Plot yk/F_0 versus $\omega_0 t$ with ω/ω_0 as a parameter

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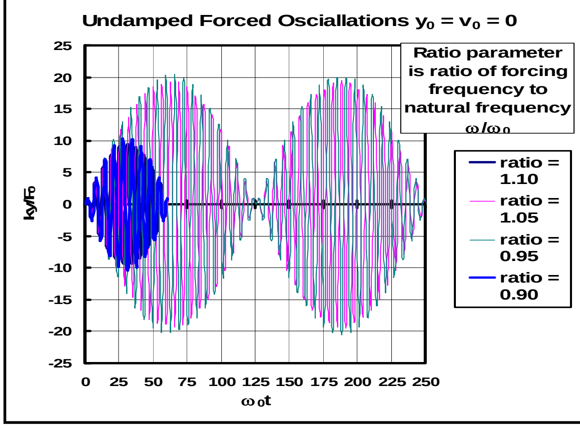

Resonance Condition

- Current equation for y/y_0 has several terms with $1 - \omega^2/\omega_0^2$ in denominator
- Solution is not valid when $\omega = \omega_0$
- If $\omega = \omega_0$, $r(x) = F_0 \cos \omega t$ is proportional to homogenous equation solution
- Have to get new particular solution
- Use undetermined coefficients approach starting with $y_P = t\,[A \sin \omega t + B \cos \omega t]$

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c = 0 Resonance Solution II

- Remember $\omega = \omega_0 = (k/m)^{1/2}$ here
- Derivatives of $y_P = t[A \sin \omega t + B \cos \omega t]$
- $y_P' = t[\omega A \cos \omega t - \omega B \sin \omega t] + A \sin \omega t + B \cos \omega t$
- $y_P'' = t[-\omega^2 A \sin \omega t - \omega^2 B \cos \omega t] + 2\omega A \cos \omega t - 2\omega B \sin \omega t$
- Substitute into ODE for c = 0 and $m = k\omega^2$: $m d^2y_P/dt^2 + k y_P = m d^2y_P/dt^2 + m\omega^2 y_P = F_0 \cos \omega t$

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c = 0 Resonance Solution III

- $m\,t[-\omega^2 A \sin \omega t - \omega^2 B \cos \omega t] + m[2\omega A \cos \omega t - 2\omega B \sin \omega t] + m\omega^2 t[A \sin \omega t + B \cos \omega t] = F_0 \cos \omega t$
- After cancellations we have $m[2\omega A \cos \omega t - 2\omega B \sin \omega t] = F_0 \cos \omega t$
- This gives B = 0 and $A = F_0/2m\omega$
- $y_P = [F_0/2m\omega]\, t \sin \omega t$
- Particular solution increases without bound as t increases

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Examine Damping

- Damped oscillations without external force
  - derived (homogenous equation) solutions last week
  - Three cases: underdamping, critical damping, and underdamping
  - All cases show y goes to zero as t increases
  - Look at particular solution only to show effect of forced oscillations
  - Effects come from amplitude of oscillations

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General Case for c ≠ 0

- Convert $y_P = C \sin \omega t + D \cos \omega t = E \cos(\omega t - \delta)$ to examine amplitude
- $C^2 + D^2 = E^2$ and $\delta = \tan^{-1}(C/D)$
- Apply this to write $y_P = E \cos(\omega t - \delta)$

$$y_P = \frac{F_0 \omega c \sin \omega t}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2} + \frac{F_0 m\left(\omega_0^2 - \omega^2\right)\cos \omega t}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}$$

$$\delta = \tan^{-1}\left(\frac{\dfrac{\omega c F_0}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}}{\dfrac{F_0 m\left(\omega_0^2 - \omega^2\right)}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}} \right) = \tan^{-1}\left(\frac{\omega c}{m\left(\omega_0^2 - \omega^2\right)} \right)$$

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Find ω that Maximizes C

$$C = \sqrt{\left(\frac{F_0 \omega c}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2} \right)^2 + \left(\frac{F_0 m\left(\omega_0^2 - \omega^2\right)}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2} \right)^2}$$

$$= F_0 \sqrt{\frac{\omega^2 c^2 + m^2\left(\omega_0^2 - \omega^2\right)^2}{\left[m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2\right]^2}} = \frac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}}$$

$$\frac{dC}{d\omega} = \frac{-\dfrac{F_0}{2}\left[m^2 2\left(\omega_0^2 - \omega^2\right)(-2\omega) + 2\omega c^2\right]}{\left[m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2\right]^{3/2}} = 0$$

$$m^2 2\left(\omega_0^2 - \omega^2\right)(-2\omega) + 2\omega c^2 = \omega\left[-4m^2\left(\omega_0^2 - \omega^2\right) + 2c^2\right] = 0$$

$$c^2 = 2m^2\left(\omega_0^2 - \omega^2\right)$$

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Amplitude of y_P versus ω

$$c^2 = 2m^2\left(\omega_0^2 - \omega^2\right) \quad \Rightarrow \quad \omega^2 = \omega_0^2 - \frac{c^2}{2m^2}$$

- Maximum amplitude equation not valid if $c^2/2m^2 > \omega_0^2 = k/m$
- Look at behavior of $y_P = C \cos(\omega t - \delta)$ by examining C versus ω
- Write dimensionless equation for C, which has dimensions of length

$$\frac{m\omega_0^2}{F_0} C = \frac{m\omega_0^2}{F_0} \frac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}}$$

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Amplitude of y_P versus ω II

$$\frac{m\omega_0^2}{F_0} C = \frac{m\omega_0^2}{F_0} \frac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}} = \frac{1}{\sqrt{\dfrac{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}{m^2 \omega_0^4}}}$$

$$\frac{m\omega_0^2 C}{F_0} = \frac{\cancel{m}kC}{\cancel{m}F_0} = \frac{1}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_0}\right)^2\right]^2 + \left(\dfrac{\omega}{\omega_0}\right)^2 \dfrac{c^2}{m^2\omega_0^2}}}$$

- Dimensionless amplitude depends on ω/ω_0 and $c^2/m^2\omega_0^2 = c^2/mk$
- Previous result: $(\omega^2/\omega_0^2)_{max} = 1 - c^2/m^2\omega_0^2$

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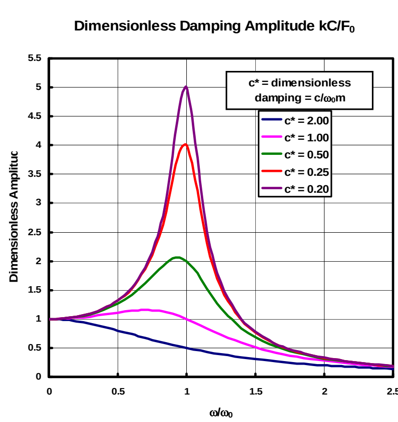

- As $c^2/m^2\omega_0^2$ becomes smaller, location of ω/ω_0 for maximum becomes one and maximum amplitude becomes $m\omega_0/c$

Summary

- General solutions for ODEs with order n ≥ 2 for constant coefficients only
  - Solutions are series of $e^{\lambda_k x}$ terms where λ_k are solutions of algebraic equation
  - Special cases: double and complex roots
- Get general solution as $y = y_H + y_P$
  - Use method of undetermined coefficients (simpler than variation of parameters) to find y_P

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